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by H. A. Hassan

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NORTH CAROLINA STATE COLLEGE
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SUMMARY

The transfer equations as given by Burgers are used to give explicit expressions for the diffusion velocities, the heat flux vectors, and the stress tensors of the constituents of a partially ionized gas. The calculation takes into consideration the presence of electric and magnetic fields and pressure, temperature, concentration, and velocity gradients. The results correspond to what is normally referred to as a "second approximation."

INTRODUCTION

The equations governing the flow of a plasma have been derived starting from the appropriate Boltzmann equations by Kolodner,[1] Burgers,[2,3] and Herdan and Liley[4] using Grad's thirteen moment method. In addition to the conservation equations, this scheme gives separate equations for the stress tensors, heat flux vectors, and the diffusion velocities. In general, the simultaneous solution of such a system of equations is quite complicated even for the simplest problems. Therefore, the more conventional approach of finding explicit expressions for the transport relations first, and using the resulting expressions in the conservation equations, is often employed. The expressions for the transport relations are usually obtained from the derived equations for the stress, diffusion, and heat flux by the method of successive approximations.

Expressions for the transport properties of a fully ionized gas were given by Burgers and Herdan and Liley. The slightly ionized case was considered by Yang.[5] In this work, the gas is assumed to be partially ionized and, to preserve symmetry, the magnetic and electric field vectors are assumed to have arbitrary directions.

The derivation is based upon Burgers' transfer equations and employs the complete equations that result in the second approximation. Because of the long-range influence of Coulomb forces, collision cross sections of charged particles are much larger than cross sections of collisions involving neutral atoms. This result and the fact that the electron-atom mass ratio is small compared to unity have been used to simplify the governing equations. Further simplification results if one assumes that the degree of ionization is not very close to zero or unity. Because these limiting cases were discussed by previous authors, this assumption will be employed here.

When the derived expressions for the stress, heat flux, and diffusion are substituted into the conservation equations, equations of the Navier-Stokes type result. Such equations can be used, among other things, for the analysis of MHD generators and plasma accelerators.

The scheme presented here for the calculation of the transport relations is an alternative to the Chapman-Enskog scheme. Experience with pure gases shows that the two schemes lead to similar results.

SYMBOLS

a_0, a_1, a_2	quantities defined by equation (40)
b_0, b_1, b_2	quantities defined by equation (42)
$\vec{B}$	magnetic field strength
C_1, C_2, C_3	quantities appearing in equation (46)
d_s	quantity defined by equation (32)
e	charge of proton
e_s	charge of species s
$e_{k\ell j}$	permutation tensor
$\vec{E}$	electric field strength

$\vec{E}^*$	$\vec{E}^* = \vec{E} + \vec{U} x \vec{B}$
f_{sh}	vector defined by equation (5)
F_h	vector defined by equation (40)
g_{sh}	vector defined by equation (6)
G_h	vector defined by equation (42)
J_{sh}	current density of species s
k	Boltzmann constant
k_s	quantity defined by equation (38)
k_{st}	friction coefficient
m_s	mass of particle s
$(M_s)_{hk}$	tensor defined by equation (8)
n	$n = n_a + n_i$, number of nuclei
n_s	number density of species s
p_s	pressure of species s
$(ps)_{hk}$	stress tensor of species s
$(P_s)_{hk}$	viscous stress tensor of species s
q_{sh}	heat flux component of species s
Q_{sh}	vector defined by equation (32)
r_{sh}	residual heat flux component of species s
R_{sh}	vector defined by equation (37)
T_s	kinetic temperature of species s
u_h	mean velocity component
$\vec{U}$	mean velocity vector
W_{sh}	diffusion velocity component of species s

$z_{st}, z'_{st}, z''_{st}$	quantities defined by equation (11)
$Z_{st}, Z_{st}^{(\ell j)}$	collision cross sections
α	degree of ionization
α_s	quantity defined by equation (62)
β_e, β_a	quantities defined by equation (26)
$\gamma, \gamma_e, \gamma_i, \gamma_a$	quantities defined by equation (30)
$\delta_e, \delta_i, \delta_a$	quantities defined by equation (26)
δ_{hk}	Kronecker delta
Δ	quantity defined by equation (64)
ϵ	$\epsilon = \partial u_h/\partial X_h$, divergence of the mean flow
ϵ_{hk}	tensor defined by equation (8)
ζ_{sh}	quantity defined by equation (11)
θ_{sh}	vector defined by equation (6)
$(\lambda_s)_{hk}$	tensors defined by equation (62)
ϑ_s	quantity defined by equation (60)
ρ	mean density
ρ_s	density of species s
$(\sigma_s)_{hk}$	tensor defined by equation (7)
τ	mean collision time between ions and electrons
Ψ_{sh}	vector defined by equation (5)
ω	electron cyclotron frequency

GOVERNING EQUATIONS

The complete equations which give the second approximation to the transport properties can be written as[2,3]

$$\sum \rho_s W_{sh} = 0 \qquad (1)$$

$$\Psi_{sh} = \sum K_{st}[(W_{sh} - W_{th}) - \frac{z_{st}}{m_s + m_t}(m_t r_{sh} - m_s r_{th})] \qquad (2)$$

$$\Theta_{sh} = \sum_{t \neq s} k\left(\frac{m_t}{m_s + m_t}\right)^2 \left(\frac{T_s}{m_s} + \frac{T_t}{m_t}\right) K_{st}\left\{\frac{5}{2} z_{st}(W_{sh} - W_{th})\right.$$
$$+ \frac{m_s}{m_s + m_t}\left[\frac{4}{5} z''_{st}(r_{sh} + r_{th}) + (3\frac{m_s}{m_t} + \frac{m_t}{m_s}\mathfrak{Z}_{st})r_{sh}\right.$$
$$\left.\left. - (3 + \mathfrak{Z}_{st})r_{th}\right]\right\} - \frac{2}{5} K_{ss}\frac{kT_s}{m_s} z''_{ss}\, r_{sh} \qquad (3)$$

$$(\sigma_s)_{hk} - (M_s)_{hk}$$
$$= -\frac{2}{5} n_s \sqrt{(kT_s/m_s)}\, Z^{(22)}_{ss} (P_s)_{kh}$$
$$- \sum_{t \neq s} Z_{st}\left(\frac{m_t}{m_s + m_t}\right)^2 \left[2k\left(\frac{T_s}{m_s} + \frac{T_t}{m_t}\right)\right]^{1/2} \left\{\left(\frac{2}{5} z''_{st} + \frac{4}{3}\frac{m_s}{m_t}\right) n_t (P_s)_{kh}\right.$$
$$\left. + n_s \frac{m_s}{m_t}\left(\frac{2}{5} z''_{st} - \frac{4}{3}\right)(P_t)_{kh}\right\}; \quad s,t = i,e,a \qquad (4)$$

where

$$\Psi_{sh} = f_{sh} + [(\vec{J}_s - \frac{\rho_s}{\rho}\vec{J}) x \vec{B}]_h; \; f_{sh} = -\frac{\partial P_s}{\partial X_h} + \frac{\rho_s}{\rho}\frac{\partial P}{\partial X_h} + n_s e_s E^*_h \qquad (5)$$

$$\Theta_{sh} = \frac{e_s P_s}{m_s}[g_{sh} - (\vec{r}_s x \vec{B})_h], \quad g_{sh} = \frac{5}{2}\frac{k}{e_s}\frac{\partial T_s}{\partial X_h} \qquad (6)$$

$$(\sigma_s)_{hk} = P_s \epsilon_{hk} - (E_h^* J_{sk} + E_k^* J_{sh} - \frac{2}{3}\delta_{hk} E_i^* J_{si}) \tag{7}$$

$$(M_s)_{hk} = \frac{e_s}{m_s}[e_{k\ell j}(P_s)_{h\ell} + e_{h\ell j}(P_s)_{k\ell}]B_j \tag{8}$$

$$\epsilon_{hk} = \frac{\partial u_k}{\partial X_h} + \frac{\partial u_h}{\partial X_k} - \frac{2}{3}\delta_{hk}\,\epsilon, \quad \epsilon = \frac{\partial u_h}{\partial X_h} \tag{9}$$

$$K_{st} = \frac{2}{3}\frac{m_s m_t}{m_s + m_t}\left[2k\left(\frac{T_s}{m_s} + \frac{T_t}{m_t}\right)\right]^{1/2} n_s n_t Z_{st} \tag{10}$$

$$z_{st} = 1 - \frac{2}{5}\frac{Z_{st}^{(12)}}{Z_{st}}, \quad z'_{st} = 1 - \frac{4}{35}\frac{Z_{st}^{(13)}}{Z_{st}}$$

$$z''_{st} = \frac{Z_{st}^{(22)}}{Z_{st}} \quad \zeta_{st} = 1 + 5z_{st} - \frac{7}{2} z'_{st} \tag{11}$$

$$r_{sh} = \frac{q_{sh}}{kn_s T_s} - \frac{5}{2} W_{sh} \tag{12}$$

$$J_{sh} = n_s e_s W_{sh}, \quad J_h = \sum J_{sh} \tag{13}$$

$$E_h^* = E_h + (\vec{U} \times \vec{B})_h \tag{14}$$

and e_s, m_s, ρ_s, n_s, T_s, P_s, W_{sh}, q_{sh}, $\vec{J}_s$, $(P_s)_{kh}$ denote, respectively, the charge, particle mass, mass density, number density, temperature, pressure, diffusion velocity, heat flux, current density, and viscous stress of species s. The quantities ρ, P, $\vec{U}$, $Z_{st}^{(\ell j)}$, $\vec{E}$, $\vec{B}$, δ_{hk}, $e_{k\ell j}$ denote the density, pressure, mean velocity, average collision cross sections, electric and magnetic fields, Kronecker delta, and the permutation tensor, respectively. The second-order tensor $(M_s)_{hk}$ was given by Burgers in component form; it can be shown, however, that it has the representation given above. In writing equation (6), the term $(e_s/m_s)E_i^*(P_s)_{hi}$ was assumed small compared to the other terms; this assumption decouples the equations for heat flow and viscous stresses and makes the solution of equations (1)-(3) independent of that of equations (4).

Before attempting a solution of equations (1)-(4) for the quantities r_{sh}, W_{sh}, and $(P_s)_{hk}$, a study of the relative orders of magnitudes of the terms on the right-hand side of equations (2)-(4) will be undertaken. The first to be considered are the friction coefficients K_{st}. Since

$$m_e << m; \quad m = m_a \approx m_i, \quad Z_{at} << Z_{st}; \quad s,t = e,i, \tag{15}$$

equations (10) and (15) show that the ratios

$$\frac{K_{et}}{K_{tt}} << 1, \quad t = a,i; \quad \frac{K_{ea}}{K_{ei}} << 1, \quad \frac{K_{ia}}{K_{ii}} << 1, \tag{16}$$

provided the degree of ionization is not very close to zero or unity.

For particles interacting according to the inverse power law; i.e., the potential energy of interaction is $\phi(r) = dr^{-\delta}$; one finds

$$z_{st} = 1 - \frac{2}{5}\, 3 - \frac{2}{\delta}, \qquad z'_{st} = 1 - \frac{4}{35}\left(4 - \frac{2}{\delta}\right)\left(3 - \frac{2}{\delta}\right)$$

$$z''_{st} = \frac{A_2(\delta)}{A_1(\delta)}\left(3 - \frac{2}{\delta}\right). \tag{17}$$

$\delta = 1$ corresponds to Coulomb interactions, $\delta = 4$ corresponds to Maxwellian molecules, while $\delta = \infty$ corresponds to hard elastic spheres. Since A_2/A_1 is of order unity, the above quantities are, in general, of order unity.

For a pure gas in the absence of electric and magnetic fields, equations (3), (4), (6), and (7) give

$$r_{sh} = -\frac{25}{4}\frac{kn_s}{K_{ss}z''_{ss}}\frac{\partial T_s}{\partial X_h},$$

$$P_s \epsilon_{hk} = -\frac{2}{5}\, n_s \sqrt{kT_s/m_s}\; Z_{ss}^{(22)} (P_s)_{hk}. \tag{18}$$

Hence,

$$\frac{r_{ih}}{r_{eh}} \approx \left(\frac{m_e}{m}\right)^{1/2}, \qquad \frac{r_{ah}}{r_{eh}} \approx \frac{\alpha}{1 - \alpha}\left(\frac{m_e}{m}\right)^{1/2} \frac{Z_{ee}}{Z_{aa}} \tag{19}$$

and

$$\frac{(P_e)_{hk}}{(P_a)_{hk}} \approx \left(\frac{m_e}{m}\right)^{1/2} \frac{Z_{aa}}{Z_{ee}}, \qquad \frac{(P_i)_{hk}}{(P_a)_{hk}} \approx \frac{Z_{aa}}{Z_{ii}}. \tag{20}$$

In writing equations (19) and (20), the results of equations (17) were employed and α is the degree of ionization.

Finally, ignoring the coupling between heat flux and diffusion, equations (1), (2), and (13) show that

$$\frac{J_{ih}}{J_h} \approx \frac{1}{1 - \alpha} \frac{K_{ia}}{K_{ie}} \approx \frac{1}{\alpha}\left(\frac{m}{m_e}\right)^{1/2} \frac{Z_{ia}}{Z_{ie}}. \tag{21}$$

The results of equations (15)-(21) are now used to simplify equations (2)-(4). Using these results, equations (1), (2), and (13) give

$$W_{ah} = -\frac{n_i}{n_a} W_{ih} = -\frac{n_i}{n_a} \frac{J_{ih}}{en_i}, \qquad W_{eh} = \frac{1}{en_e}(J_{ih} - J_h) \tag{22}$$

$$\Psi_{eh} = -K_{ie} \frac{J_h}{en_e} - z_{ie}K_{ie}r_{eh} \tag{23}$$

$$\Psi_{ah} = -K_{ai}\left(1 + \frac{n_i}{n_a}\right)\frac{J_{ih}}{en_i} + z_{ea}K_{ea}r_{eh} - \frac{1}{2} z_{ai}K_{ai}r_{ah}. \tag{24}$$

Similarly, equations (3) and (4) yield, respectively,

$$\theta_{eh} = -\delta_e(r_{eh} + \beta_e J_h), \quad \theta_{ih} = -\delta_i r_{ih},$$

$$\theta_{ah} = -\delta_a(r_{ah} + \beta_a J_{ih}), \tag{25}$$

where

$$\delta_e = \frac{kT_e}{m_e}\left(\frac{2}{5} K_{ee} z''_{ee} + K_{ei} \mathcal{J}_{ei}\right), \qquad \delta_i = \frac{2}{5} \frac{kT_i}{m} K_{ii} z''_{ii}$$

$$\delta_a = \frac{k}{m}\left\{\frac{2}{5} T_a K_{aa} z''_{aa} + \frac{K_{ai}}{4}(T_a + T_i)\left[\frac{2}{5} z''_{ai} + \frac{1}{2} (3 + \mathcal{J}_{ai})\right]\right\}$$

$$\beta_e = \frac{5}{2} \frac{kT_e}{m_e} \frac{K_{ie} z_{ie}}{en_e \delta_e}, \qquad \beta_a = \frac{5}{8} \frac{k(T_a + T_i)}{m} \frac{K_{ai} z_{ai}}{en_i \delta_a}\left(1 + \frac{n_i}{n_a}\right) \tag{26}$$

and

$$(\sigma_e)_{hk} - (M_e)_{hk} = - \gamma_e (P_e)_{hk} \tag{27}$$

$$(\sigma_i)_{hk} - (M_i)_{hk} = - \gamma_i (P_i)_{hk} - \gamma (P_a)_{hk} \tag{28}$$

$$p_a \epsilon_{hk} = - \gamma_a (P_a)_{hk}, \tag{29}$$

where

$$\gamma_e = \frac{2}{5}(1 + \sqrt{2}) n_e \sqrt{kT_e/m_e}\, Z_{ee}^{(22)}$$

$$\gamma_i = \frac{2}{5} n_i \sqrt{kT_i/m}\, Z_{ii}^{(22)}$$

$$\gamma_a = \frac{2}{5} n_a \sqrt{kT_a/m}\, Z_{aa}^{(22)} + \frac{1}{4} n_i Z_{ai}\left[\frac{2k}{m}(T_i + T_a)\right]^{1/2}\left(\frac{2}{5} z''_{ai} + \frac{4}{3}\right)$$

$$\gamma = \frac{1}{4} n_i Z_{ai}\left[\frac{2k}{m}(T_i + T_a)\right]^{1/2}\left(\frac{4}{3} - \frac{2}{5} z''_{ai}\right). \tag{30}$$

Solution of the present problem reduces, therefore, to the solution of the system (22)-(29). Equations (22)-(25) are solved first for r_{sh} and J_{sh}. The resulting expressions for J_{sh} are then used in equations (27)-(29) to give the desired expressions for $(P_s)_{hk}$.

The Diffusion Velocities and Heat Flux Vectors

The diffusion velocities and the heat flux vectors can be calculated from the simultaneous solutions of equations (22)-(25). The method to be employed in solving this system is to use equations (6) and (25) to express r_{sh} in terms of J_{sh} (or W_{sh}) and then employ the resulting expressions in equations (22)-(24) to give explicit expressions for J_{sh}.

It is seen from equations (6) and (25) that the equations for r_{sh} can be written as

$$r_{sh} - d_s(\vec{r}_s x \vec{B})_h = Q_{sh}, \tag{31}$$

where

$$d_s = \frac{e_s P_s}{m_s \delta_s}, \quad Q_{eh} = - d_e g_{eh} - \beta_e J_h, \quad Q_{ih} = - d_i g_{ih}$$

$$Q_{ah} = - \frac{5}{2} \frac{kP_a}{m\delta_a} \frac{\partial T_a}{\partial X_h} - \beta_a J_{ih}. \tag{32}$$

The solution of equation (31) can be expressed as

$$r_{sh} = \frac{1}{1 + (Bd_s)^2}[Q_{sh} + d_s(\vec{Q}_s x \vec{B})_h + d_s^2(\vec{Q}_s \cdot \vec{B})B_h]. \tag{33}$$

Hence,

$$r_{eh} = R_{eh} - \frac{\beta_e}{1 + (Bd_e)^2}[J_n + d_e(\vec{J} x \vec{B})_h + d_e^2(\vec{J} \cdot \vec{B})B_h] \tag{34}$$

$$r_{ih} = R_{ih} \tag{35}$$

$$r_{ah} = R_{ah} - \beta_a J_{ih}, \tag{36}$$

where

$$R_{sh} = \frac{-d_s}{(1 + (Bd_s)^2}[g_{sh} + d_s(\vec{g}_s x\vec{B})_h + d_s^2(\vec{g}_s \cdot \vec{B})B_h]$$

$$= - k_s\left[\frac{\partial T_s}{\delta X_s} + d_s(\nabla T_s x\vec{B})_h + d_s^2(\nabla T_s \cdot \vec{B})B_h\right] \quad (37)$$

and

$$k_s = \frac{5kP_s}{2m_s\delta_s[1 + (Bd_s)^2]} . \quad (38)$$

Equations (34) and (36) are now substituted into equation (24). After rearrangement, one finds

$$J_{ih} = F_h + a_1[J_h + d_e^2(\vec{J} \cdot \vec{B})B_h] + a_2(\vec{J}x\vec{B})_h, \quad (39)$$

where

$$F_h = \frac{1}{a_0}[-f_{ah} + z_{ea}K_{ea}R_{eh} - \frac{z_{ai}}{2} K_{ai}R_{ah}]$$

$$a_1 = - \frac{\beta_e z_{ea} K_{ea}}{a_0[1 + (Bd_e)^2]}, \qquad a_2 = \frac{1}{a_0}\frac{n_a}{n} + a_1 d_e$$

$$a_0 = K_{ai}\left[\frac{1}{en_i}\left(1 + \frac{n_i}{n_a}\right) - \frac{1}{2} z_{ai}\beta_a\right], \qquad n = n_a + n_i \quad (40)$$

Similarly, equations (5), (23), and (39) give

$$J_h = G_h + b_1(\vec{J}x\vec{B})_h + b_2(\vec{J} \cdot \vec{B})B_h, \quad (41)$$

where

$$G_h = \frac{1}{b_0}[f_{eh} - (\vec{F}x\vec{B})_h + z_{ie}K_{ie}R_{eh}]$$

$$b_1 = \frac{1}{b_0}\left[1 - \frac{z_{ie}K_{ie}\beta_e d_e}{1 + (Bd_e)^2}\right], \quad b_2 = -\frac{1}{b_0}\left[a_2 + \frac{z_{ie}K_{ie}\beta_e d_e^2}{1 + (Bd_e)^2}\right]$$

$$b_0 = -a_2B^2 - \frac{K_{ie}}{en_e} + \frac{z_{ie}K_{ie}\beta_e}{1 + (Bd_e)^2} . \tag{42}$$

Equation (41) can be solved to give an explicit expression for J_h. The result may be expressed as

$$J_h = \frac{1}{1 + (b_1B)^2}[G_h + b_1(\vec{G}x\vec{B})_h + \frac{b_2 + b_1^2}{1 - b_2B^2}(\vec{G}\cdot\vec{B})B_h]. \tag{43}$$

If the effects of gradients are negligible, equation (43) may be used to give an expression for the electrical conductivity.[6]

As a first step in writing expressions for the heat flux vectors and the diffusion velocities, expressions for F_h and G_h in terms of the pressure, temperatures' and concentrations' gradients, and the electric and magnetic fields will be given. Using equations (5), (37), and (40), one obtains

$$F_h = \frac{1}{a_0}\left\{\frac{\partial P_a}{\partial X_h} - \frac{n_a}{n}\frac{\partial P}{\partial X_h} + \frac{1}{2} z_{ai}K_{ai}k_a \frac{\partial T_a}{\partial X_h} - z_{ea}K_{ea}k_e\left[\frac{\partial T_e}{\partial X_h}\right.\right.$$
$$\left.\left. + d_e(\nabla T_e x\vec{B})_h + d_e^2(\nabla T_e\cdot\vec{B})B_h\right]\right\}. \tag{44}$$

Similarly, equations (5), (16), (37), and (42) give

$$G_h = -\frac{1}{b_0}\left\{\frac{\partial P_e}{\partial X_h} + en_eE_h^* + \frac{1}{a_0}[\nabla P_a - \frac{n_a}{n}\nabla P + \frac{1}{2} z_{ai}K_{ai}\nabla T_a)x\vec{B}]_h\right.$$
$$\left. + z_{ie}K_{ie}k_e\left[\frac{\partial T_e}{\partial X_h} + d_e(\nabla T_e x\vec{B})_h + d_e^2(\nabla T_e\cdot\vec{B})B_h\right]\right\} . \tag{45}$$

It is seen from equations (34), (36), and (39) that a vector of the form

$$C_1\vec{J} + C_2(\vec{J}x\vec{B}) + C_3(\vec{J}\cdot\vec{B})\vec{B} \tag{46}$$

appear frequently. Using equations (43) and (45), it can be shown that the above vector can be represented as

$$\frac{1}{1 + (b_1 B)^2}\Big\{(C_1 - b_1 C_2 B^2)\vec{G} + (b_1 C_1 + C_2)\vec{G}x\vec{B}$$

$$+ \left[\frac{C_1(b_2 + b_1^2) + C_3(1 + b_1^2 B^2)}{1 - b_2 B^2} + b_1 C_2\right](\vec{G}\cdot\vec{B})\vec{B}\Big\}, \tag{47}$$

where

$$\vec{G}x\vec{B} = -\frac{1}{b_0}\Big\{(\nabla P_e + en_e\vec{E}^*)x\vec{B} + \frac{1}{a_0}[(\nabla P_a - \frac{n_a}{n}\nabla P$$

$$+ \frac{1}{2} z_{ai}K_{ai}\nabla T_a)\cdot\vec{B}]\vec{B} - \frac{B^2}{a_0}(\nabla P_a - \frac{n_a}{n}\nabla P + \frac{1}{2} z_{ai}K_{ai}\nabla T_a)$$

$$+ \frac{z_{ie}K_{ie}k_e}{d_e}[(Bd_e)^2\nabla T_e - d_e(\nabla T_e x\vec{B}) - d_e^2(\nabla T_e\cdot\vec{B})\vec{B}] \tag{48}$$

and

$$\vec{G}\cdot\vec{B} = -\frac{1}{b_0}[\nabla P_e + en_e\vec{E}^* + z_{ie}K_{ie}k_e(1 + d_e^2 B^2)\nabla T_e]\cdot\vec{B}. \tag{49}$$

The derived expressions for r_{sh} and J_{sh} can now be used to express W_{sh} and q_{sh}. Using equations (22) and (39), one finds

$$W_{ah} = -\frac{n_i}{n_a} W_{ih} = -\frac{1}{en_a}\Big\{F_h + a_1[J_h + d_e^2(\vec{J}\cdot\vec{B})B_h]$$

$$+ a_2(\vec{J}x\vec{B})_h\Big\} \tag{50}$$

and

$$W_{eh} = \frac{1}{en_e}[F_h + (a_1 - 1)J_h + a_2(\vec{J}x\vec{B})_h + a_1 d_e^2(\vec{J}\cdot\vec{B})B_h]. \qquad (51)$$

Similarly, the heat flux vectors are obtained from equations (12) and (34)-(36) as

$$q_{eh} = P_e\Big\{R_{eh} - \frac{\beta_e}{1 + (Bd_e)^2}[J_h + d_e(\vec{J}x\vec{B})_h$$

$$+ d_e^2(\vec{J}\cdot\vec{B})B_h] + \frac{5}{2} W_{eh}\Big\} \qquad (52)$$

$$q_{ih} = P_i[R_{ih} + \frac{5}{2} W_{ih}] \qquad (53)$$

and

$$q_{ah} = P_a\Big\{R_{ah} - \beta_a\Big\{F_h + a_1[J_h + d_e(\vec{J}\cdot\vec{B})B_h]$$

$$+ a_2(\vec{J}x\vec{B})_h\Big\} + \frac{5}{2} W_{ah}\Big\}. \qquad (54)$$

The total heat flux vector, q_h, is obtained by adding equations (52)-(54). The results may be written as

$$q_h = \sum P_s R_{sh} + \frac{5}{2}\sum P_s W_{sh} - \beta_a P_a J_{ih}$$

$$- \frac{\beta_e P_e}{1 + (Bd_e)^2}[J_h + d_e(\vec{J}x\vec{B})_h + d_e^2(\vec{J}\cdot\vec{B})B_h]. \qquad (55)$$

The explicit dependence of the above vector on pressure, temperatures and concentrations' gradients, and electric and magnetic fields may be obtained by utilizing equations (46) and (47). In general, each of the vectors under discussion may be represented as

$$\sum a_s \nabla P_s + \sum b_s \nabla T_s + \sum C_s (\nabla P_s x \vec{B}) + \sum f_s (\nabla T_s x \vec{B})$$

$$+ g_s \vec{B} + h_s \vec{E}^* + W_s (\vec{E}^* x \vec{B}); \quad s = a,i,e, \tag{56}$$

with

$$\nabla P_s = k n_s \nabla T_s + k T_s \nabla n_s. \tag{57}$$

The substitutions into equations (50)-(55) are straightforward. However, because the resulting expressions are unusually long, they will not be included here.

The above expressions assume a relatively simple form if one assumes that collisions with the atoms follow the Maxwellian molecule approximation. In this case,

$$z_{at} = z'_{at} = 0, \quad z''_{at} = 2.59, \quad \zeta_{at} = 1, \quad t = a,e,i. \tag{58}$$

For charged particles $z_{st} = 3/5$, $z''_{st} = 2$, $\zeta_{st} = 1.3$, and, therefore,

$$d_s B = \omega\tau/\nu_s, \tag{59}$$

where

$$\omega = eB/m_e, \quad \tau = n_e m_e/K_{ie}, \quad \nu_e = -1.865$$

$$\nu_i = K_{ii}/K_{ie}. \tag{60}$$

For the case where $T_e = T_i$, $\nu_i = (m/2m_e)^{1/2}$.

The Stress Tensors

The stress tensors can be calculated from equations (27)-(29). Equation (29) gives the desired expression for the neutral viscous stress tensor and shows that the second approximation to $(P_a)_{kh}$ is not influenced by the electric and magnetic fields. Using equations (8), (27)-(29), it is seen that $(P_s)_{kh}$ is governed by an equation of the type

$$(P_s)_{hk} + \alpha_s[e_{k\ell j}(P_s)_{h\ell} + e_{h\ell j}(P_s)_{k\ell}]B_j = (\lambda_s)_{kh}, \quad s = i,e,a, \tag{61}$$

where

$$\alpha_s = -\frac{e_s}{m_s\gamma_s}, \qquad (\lambda_e)_{hk} = -\frac{(\sigma_e)_{hk}}{\gamma_e}, \qquad (\lambda_a)_{hk} = -\frac{P_a}{\gamma_a}\epsilon_{hk}$$

$$(\lambda_i)_{hk} = -\frac{(\sigma_i)_{hk}}{\gamma_i} + \frac{\gamma}{\gamma_i}\frac{P_a}{\gamma_a}\epsilon_{hk}. \tag{62}$$

The solution of equation (61) can be written as

$$\begin{aligned}
(P_s)_{hk}
&= \frac{1}{D_s}\Big\{(\lambda_s)_{kh} + \alpha_s[e_{jk\ell}(\lambda_s)_{hj} + e_{\ell jk}(\lambda_s)_{kj}]B_\ell \\
&+ 2\alpha_s^2[(\delta_{hk}B^2 - B_hB_k)(\lambda_s)_{jj} - \delta_{hk}B_\ell B_j(\lambda_s)_{\ell j}] \\
&+ \frac{3}{\Delta_s}\Big\{\alpha_s^2[B_h(\lambda_s)_{km} + B_k(\lambda_s)_{hm}]B_m \\
&+ \alpha_s^3(e_{h\ell j}B_k + e_{k\ell j}B_h)B_mB_\ell(\lambda_s)_{mj} + 2\alpha_s^4 B_kB_hB_jB_\ell(\lambda_s)_{j\ell}\Big\}\Big\},
\end{aligned} \tag{63}$$

where

$$D_s = 1 + 4(\alpha_sB)^2, \qquad \Delta_s = 1 + (\alpha_sB)^2. \tag{64}$$

Assuming $T_e = T_i$ and using equations (30), one finds

$$\alpha_eB = \frac{5}{3(2+\sqrt{2})}\,\omega\tau, \qquad \alpha_iB = -\frac{5}{2}\left(\frac{m_e}{2m}\right)^{1/2}\omega\tau. \tag{65}$$

The expressions given above for $(P_i)_{kh}$ and $(P_e)_{kh}$ depend on the current densities J_{sh}. With J_{ih} and J_n given by equations (39) and (43), equations (63) give the desired expressions for the viscous stress tensors.

The over-all stress tensor p_{hk} is defined as

$$p_{hk} = \delta_{hk} \sum p_s + \sum (p_s)_{hk} = \delta_{hk} p + \sum (p_s)_{hk}. \qquad (66)$$

p_{hk} may be calculated from equation (63). For small electric fields it is expected that equations (20) will hold, at least approximately. In such cases, the over-all viscous stress tensor may be approximated by the neutral viscous stress tensor.

Discussion and Conclusions

Explicit expressions for the transport relations of a partially ionized gas in the presence of electric and magnetic fields and pressure, temperatures, concentrations, and velocity gradients are derived. The derivation assumes that the plasma is neutral, the degree of ionization is not very close to zero or unity, the collision cross sections of charged particles are much larger than cross sections involving neutral atoms, and $m_e/m << 1$.

The first assumption is probably the most restrictive because the plasma is not neutral in the sheath region. However, removing such a restriction would result in expressions which are much more complex than those presented here. Therefore, in spite of the fact that this case can be handled by the method presented here, it was thought that such a refinement was not warranted at present.

The second and third assumptions are actually interrelated. The degree of ionization should be such that equations (16) are satisfied. For degrees of ionization smaller than the lower limit, the results for the slightly ionized case hold, while for values greater than the maximum, the gas may be treated as fully ionized.

The question arises whether the derived expressions can explain the presence of the so-called anomalous or turbulent diffusion. Keeping other things constant, the expression for W_{sh} shows that for large B the diffusion is proportional to

$1/B^2$. This is the prediction of the so-called classical diffusion. However, in cases like discharges crossed with magnetic fields where anomalous or $1/B$ diffusion has been observed, the conservation equations show that the flow properties depend, in a given situation, on the power input and the magnetic field. This means that if one takes the variation of the degree of ionization, temperature, etc., with B, there is the possibility that $1/B$ diffusion can be observed over a certain range of operating conditions.

The equations presented here are not valid in the range where $\tau \rightarrow \infty$. This is because equations (2)-(4) are not valid for a collisionless plasma. To obtain meaningful results in the limit of $\tau \rightarrow \infty$, the exact equations should be employed.

The derived expressions for q_{sh}, W_{sh}, $(P_s)_{kh}$ can be used to derive expressions for the transport coefficients. Because $T_e \neq T_i \neq T_a$, one can calculate, in addition to the usual coefficients, "multicomponent heat conduction coefficients." Such a calculation is not reported here because the conservation equations are expressed in terms of the fundamental quantities W_{sh}, J_{sh}, q_{sh}, and $(P_s)_{kh}$ and not the transport coefficients.

North Carolina State
Raleigh, North Carolina
February 28, 1964

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